

# A REMARK ON LIAO AND RAMS' RESULT ON DISTRIBUTION OF THE LEADING PARTIAL QUOTIENT WITH GROWING SPEED $e^{n^{1/2}}$ IN CONTINUED FRACTIONS

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ABSTRACT. For a real  $x \in (0, 1) \setminus \mathbb{Q}$ , let  $x = [a_1(x), a_2(x), \dots]$  be its continued fraction expansion. Denote by

$$T_n(x) := \max\{a_k(x) : 1 \leq k \leq n\}$$

the leading partial quotient up to  $n$ . For any real  $\alpha \in (0, \infty), \gamma \in (0, \infty)$ , let

$$F(\gamma, \alpha) := \{x \in (0, 1) \setminus \mathbb{Q} : \lim_{n \rightarrow \infty} \frac{T_n(x)}{e^{n^\gamma}} = \alpha\}.$$

For a set  $E \subset (0, 1) \setminus \mathbb{Q}$ , let  $\dim_H E$  be its Hausdorff dimension. Recently Lingmin Liao and Michal Rams [LR, Theorem 1.3] show that

$$\dim_H F(\gamma, \alpha) = \begin{cases} 1 & \text{if } r \in (0, 1/2) \\ 1/2 & \text{if } r \in (1/2, \infty) \end{cases}$$

for any  $\alpha \in (0, \infty)$ . In this paper we show that  $\dim_H F(1/2, \alpha) = 1/2$  for any  $\alpha \in (0, \infty)$  following Liao and Rams' method, which supplements their result.

Through out the paper we follow Liao and Rams' notations [LR]. As mentioned in the abstract, we aim to show that

## Theorem 1.

$$\dim_H F(1/2, \alpha) = 1/2.$$

We only prove  $\dim_H F(1/2, 1) = 1/2$ , as one can show the theorem for any  $\alpha \in \mathbb{R}^+ := (0, \infty)$  by the same process. In order to do this, we first show that

## Lemma 1.

Let  $L \in \mathbb{R}^+$  be a constant. Let  $n_k := [(\frac{k}{L})^2]$  (the integer part of  $(\frac{k}{L})^2$ ),  $k \in \mathbb{N}$ . Then for any  $x \in F(1/2, 1)$  and  $k$  large enough, there exists an integer  $j_k, n_{k-1} < j_k \leq n_k$ , such that

$$T_{n_k}(x) = a_{j_k}(x).$$

*Proof.* We prove this by reduction to absurdity. Suppose there exist infinitely many integers  $k_i, j_{k_i}, i \in \mathbb{N}, k_i > k_{i-1}, j_{k_i} \leq n_{k_i-1}$ , such that

$$T_{n_{k_i}}(x) = a_{j_{k_i}}(x)$$

for some  $x \in F(1/2, 1)$ . Note that in this case we have

$$T_{n_{k_i-1}}(x) = a_{j_{k_i}}(x).$$

Then for the sequence  $\{n_{k_1-1}, n_{k_2-1}, \dots\}$ , we have

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$$\lim_{i \rightarrow \infty} \frac{T_{n_{k_i-1}}(x)}{e^{\frac{1}{2}n_{k_i-1}}} = \lim_{i \rightarrow \infty} \frac{T_{n_{k_i}}(x)}{e^{[(k_i-1)^2/L^2]^{1/2}}} = \lim_{i \rightarrow \infty} \frac{T_{n_{k_i}}(x)}{e^{\frac{1}{2}n_{k_i}}} \frac{e^{[k_i^2/L^2]^{1/2}}}{e^{[(k_i-1)^2/L^2]^{1/2}}} = 1 \cdot e^{1/L} \neq 1$$

which contradicts the fact that

$$\lim_{k \rightarrow \infty} \frac{T_k(x)}{e^{k^{1/2}}} = 1$$

as  $x \in F(1/2, 1)$ . So our conclusion holds for any sufficiently large  $k$ . ■

In the following we will omit the integer notation  $[\ ]$  for simplicity as results will not be affected. By this lemma,

**Corollary 1.**

For  $x \in F(1/2, 1)$  and  $n_k := (\frac{k}{L})^2$ , we have

$$(1 - \epsilon)e^{k/L} \leq S_{n_k}(x) - S_{n_{k-1}}(x) \leq (1 + \epsilon)(\frac{k}{L})^2 e^{k/L}$$

for a small  $\epsilon \in \mathbb{R}^+$  and any  $k$  large enough.

The rest of the work goes the same process as estimation of the upper bound for  $E_\varphi$  when  $\gamma = 1/2$  in [LR, Proof of Theorem 1.1]. For the length of the rank- $n$  fundamental interval

$$I_n(a_1, \dots, a_n) := \{x \in (0, 1) \setminus \mathbb{Q} : a_1(x) = a_1, \dots, a_n(x) = a_n\},$$

we have

$$\prod_{i=1}^n \frac{1}{(a_i+1)^2} I_n(a_1, \dots, a_n) \leq \prod_{i=1}^n \frac{1}{a_i^2}.$$

Let

$$A(m, n) := \{(i_1, \dots, i_n) \in \{1, \dots, m\}^n : \sum_{j=1}^n i_j = m\}.$$

Let  $\zeta(\cdot)$  be the Riemann zeta function. Now we quote [LR, Lemma 2.1] as following.

**Lemma 2.**

For  $s \in (1/2, 1)$  and  $m \geq n$ , we have

$$\sum_{(i_1, \dots, i_n) \in A(m, n)} \prod_{j=1}^n \frac{1}{i_j^{2s}} \leq \left(\frac{9}{2}(2 + \zeta(2s))\right)^n \frac{1}{m^{2s}}.$$

Now we are in a position to bound Hausdorff dimension of  $F(1/2, 1)$  above.

**Theorem 2.**

$\dim_H F(1/2, 1) \leq 1/2$ .

*Proof.* Let  $D_l$  be the integers in the interval  $[(1 - \epsilon)e^{l/L}, (1 + \epsilon)(\frac{l}{L})^2 e^{l/L}]$ . Let

$$B(1/2, N) := \{\cup_{k=N}^\infty I_{n_k}(a_1, a_2, \dots, a_{n_k}) : \sum_{j=n_{l-1}+1}^{n_l} a_j = m \text{ with } m \in D_l, N \leq l \leq k\}.$$

By Corollary 1 one can see that

$$F(1/2, 1) \subset \cup_{N=1}^\infty B(1/2, N).$$

Now we show that  $\dim_H B(1/2, 1) \leq 1/2$ . Similar method implies  $\dim_H B(1/2, N) \leq 1/2$  for any  $N \in \mathbb{N}$ , which is enough to prove our Theorem 2. By Lemma 2,

$$\sum_{I_{n_k} \subset B(1/2, 1)} |I_{n_k}|^s \leq \prod_{l=1}^k \sum_{m \in D_l} \left(\frac{9}{2}(2 + \zeta(2s))\right)^{n_l - n_{l-1}} \frac{1}{m^{2s}}.$$

Note that  $|D_l| \leq (1 + \epsilon)(\frac{k}{L})^2 e^{k/L}$ ,  $m > (1 - \epsilon)e^{k/L}$ , so

$$\begin{aligned} & \sum_{I_{n_k} \subset B(1/2, 1)} |I_{n_k}|^s \\ & \leq \prod_{l=1}^k (1 + \epsilon)(1 - \epsilon)^{2s} (l/L)^2 e^{(1-2s)l/L} \left(\frac{9}{2}(2 + \zeta(2s))\right)^{\frac{2l-1}{L^2}} \\ & \leq \prod_{l=1}^k \left( ((1 + \epsilon)(1 - \epsilon)^{2s} (l/L)^2)^{1/l} e^{(1-2s)/L} \left(\frac{9}{2}(2 + \zeta(2s))\right)^{3/L^2} \right)^l. \end{aligned}$$

Solve the equation

$$\frac{9}{2}(2 + \zeta(2s)) = \frac{1}{2} e^{\frac{2s-1}{3}L}$$

regarding the main terms, we get a unique solution  $s_L \in (1/2, 1)$  when  $L$  is large enough.  $s_L \rightarrow 1/2$  as  $L \rightarrow \infty$  since  $\zeta(2 \cdot \frac{1}{2}) = \zeta(1) = \infty$ . Then  $\sum_{I_{n_k} \subset B(1/2, 1)} |I_{n_k}|^s < \infty$ , which forces  $\dim_H B(1/2, 1) \leq 1/2$ . ■

As  $\dim_H F(1/2, \alpha) \geq 1/2$  (see [LR, Proof of Theorem 1.3]), Theorem 1 follows directly from Theorem 2.

**Remark 1.**

*Our corollary 1 sharpens estimation on  $S_{n_k}(x) - S_{n_{k-1}}(x)$  in [LR, Proof of Theorem 1.3] for  $x \in F(1/2, 1)$ . In fact we can do similar estimations for any  $x \in F(\gamma, \alpha)$ ,  $\gamma \in (0, \infty)$ ,  $\alpha \in \mathbb{R}^+$ ,  $n_k = k^{1/\gamma}$ . This enables us to give better estimation on  $\sum_{I_{n_k} \subset B(\gamma, N)} |I_{n_k}|^s$ ,  $\gamma \in [1/2, 1)$ . By virtue of it, when estimating the upper bound in [LR, Proof of Theorem 1.3] for  $H$ -dimension of  $F(\gamma, \alpha)$ ,  $\gamma \in (1/2, 1)$ , we can simply take  $n_k = k^{1/\gamma}$  instead of  $k^{1/\gamma}(\log k)^{1/\gamma^2}$ .*

## REFERENCES

- [LR] Lingmin Liao and Michal Rams, Subexponentially increasing sums of partial quotients in continued fraction expansions, Math. Proc. Cambridge Philos. Soc. 160, no. 03, 401-412, 2016.

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